

Three dimensional magnetic solutions in massive gravity with (non)linear field

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The Noble Prize in physics 2016 motivates one to study different aspects of topological properties and topological defects as their related objects. Considering the significant role of the topological defects (especially magnetic strings) in cosmology, here, we will investigate three dimensional horizonless magnetic solutions in the presence of two generalizations: massive gravity and nonlinear electromagnetic field. The effects of these two generalizations on properties of the solutions and their geometrical structure are investigated. The differences between de Sitter and anti de Sitter solutions are highlighted and conditions regarding the existence of phase transition in geometrical structure of the solutions are studied.

I. INTRODUCTION

The Nobel Prize in physics 2016 has been assigned to the interesting consequences of topological invariant and topological phase transitions. Although general relativity is based on the local transformation, it is found that topological properties have significant impacts on our understanding of the universe. In this regard, topological defects have been observed in the various branches of physics. The crucial role of topological defects was observed in a new type of phase transition in two-dimensional systems [1, 2]. Kosterlitz and Thouless applied the mentioned points to superconducting and superfluid films and they found their important roles in quantum nature of one-dimensional systems at very low temperatures. In addition, it was shown that phenomenological properties of different phases of physical systems could be explained by these topological defects. For example, in studying liquid crystal, it was shown that structural properties and phase transitions are affected by topological effects [3–5]. The applications of these defects in condensed matter with ordered media [6], magnetism and nanomagnetism [7, 8], vortices in superfluid [9] and Bose-Einstein condensate [10–12] were explored. Furthermore, the importance of these mathematical tools in studying superconductors and their phase transitions were highlighted in Refs. [13–16].

From the cosmological point of view, existence of the topological defects could be traced back into early universe and also phase transitions in the early universe [17–23]. The existence of topological defects is originated from breaking down the symmetry in phase transitions that has taken place in the early universe. Speaking more precisely, regions of the universe which are separated by more than the distance $d = ct$ (in which c is speed of light and t is time), could not know anything about each other. During the phase transition, different regions choose different minima in the set of possible states to fall in. The topological defects are placed at the boundaries of these regions which have chosen different minima. Therefore, one can state that topological defects are the results of disagreement between different choices of different regions. In the context of cosmology, there are different types of the topological defects. Depending on the dimensionality and structural properties, these defects could be categorized into; (i) Domain walls which are due to a broken discrete symmetry and divide universe into blocks. (ii) Cosmic strings which are due axial or cylindrical symmetry breaking and are related to grand unified particle physics models/electroweak scale. (iii) Monopoles which are super massive and carry magnetic charge and are formed when a spherical symmetry is broken. (iv) Textures which are formed due to the breaking of several symmetries. These topological defects carry information regarding the early universe. In addition, it was proposed that they could have specific roles in the large-scale structures [17–23], anisotropy in the Cosmic Microwave Background (CMB) [24–26] and dark matter [27–30]. Besides, these topological defects could be used as cosmological lenses [31]. In other words, the trajectory of the photon on these topological defects are affected depending on deficit angle. This highlights the importance of analyzing deficit angle in the properties of topological defects.

The cosmic strings in the presence of Maxwell field have been investigated [32–34]. Furthermore, the superconducting property of these topological defects has been explored in Einstein [35–37], dilaton [38–40] and Brans-Dicke [41] theories. In addition, the QCD applications of the magnetic strings [42–44] and their roles in quantum theories [45–47] have been investigated before. The stability of the cosmic strings through quantum fluctuations has been analyzed

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in Ref. [48]. The limits on the cosmic string tension have been studied by extracting signals of cosmic strings from CMB temperature anisotropy maps [49]. The spectrum of gravitational wave background produced by cosmic strings is obtained in Ref. [50]. For further investigations regarding cosmic strings, we refer the reader to an incomplete list of references [51–59].

Domain walls and their evolution in de Sitter universe have been studied in [60]. In addition, the gravitational waves produced from decaying domain walls are investigated in Ref. [61]. The localization of the fields on the dynamical domain wall was investigated and it was shown that the chiral spinor can be localized on the domain walls [62]. For further studies regarding this class of topological defects, we refer the reader to Refs. [63–71].

On the other hand, considering most of physical systems in nature, one finds that they exhibit nonlinear behavior, and therefore, the nonlinear field theories are of importance in physical researches. There are many motivations for studying the nonlinear electrodynamics (NED) such as; (i) These theories are the generalizations of Maxwell field and reduce to linear Maxwell theory in the special cases (weak nonlinearity). (ii) These nonlinear theories can describe the radiation propagation inside specific materials [72–75]. (iii) Some special NED models can describe the self-interaction of virtual electron-positron pairs [76–78]. (iv) Theories of NED can remove the problem of point-like charge self-energy. (v) From the standpoint of quantum gravity and its coupling with these nonlinear theories, we can obtain more information and deep insight regarding the nature of gravity [79]. (vi) Compatibility with AdS/CFT correspondence and string theory are other properties of NED theories. (vii) NED theory improves the basic concept of gravitational redshift and its dependency of any background magnetic field as compared to the well-established method introduced by general relativity. (viii) From the perspective of cosmology, it was shown that NED theories can remove both of the big bang and black hole singularities [80–84]. (ix) From astrophysical point of view, it was found that the effects of NED become indeed quite important in super-strong magnetized compact objects, such as pulsars and particular neutron stars [85–87].

There are different models of NED, such as Born-Infeld form [88], logarithmic form [89], exponential form [90], arcsin-electrodynamics form [91] and etc. One of the interesting branches of the nonlinear electrodynamics is power Maxwell invariant (PMI) theory. The Lagrangian of PMI theory is an arbitrary power of the Maxwell Lagrangian [92–95] which could reduce to the Maxwell field by choosing the unit power. In addition, the PMI theory has an interesting consequence which distinguishes this NED theory from other theories; this theory enjoys conformal invariance when the power of Maxwell invariant is a quarter of space-time dimensions (power = dimensions/4). In other words, in this case, the energy-momentum tensor will be a traceless tensor which leads to conformal invariance and also an inverse square law of the electric field for the point-like charge in arbitrary dimensions [92, 96].

Recent observations of gravitational waves from a binary black hole merger in LIGO and Virgo collaboration provided a deep insight to general relativity (GR) and existence of massive gravitons [97]. However, gravitons are massless particles with spin 2 in GR which have two degrees of freedom. Since the quantum theory of massless gravitons is non-renormalizable [98], in order to remove this problem, one may modify GR to massive gravity by adding a mass term to the Einstein-Hilbert action. Therefore, considering this action, the graviton will have a mass of m which in case of $m \rightarrow 0$, the effect of massive gravity will be vanished. In other words, massive gravity is a modification of GR that gravitons have mass. Among the motivations of the massive gravity one can mention description of accelerating expansion of universe without considering the cosmological constant [99, 100]. It was shown that the terms of massive gravitons can be equivalent to a cosmological constant [101–103]. This theory modifies gravity compared with GR which allows the universe to accelerate at the large scale, however at small scale, this theory reduces to GR as well. This theory of gravity may illustrate the dark energy problem [104, 105]. In addition, the existence of massive gravitons provides extra polarization for gravitational waves, and affects the propagation's speed of the gravitational waves [106], hence, the production of gravitational waves during inflation [107, 108]. By adding the interaction terms to GR, massive gravity with flat background was investigated by Fierz and Pauli [109]. However, this theory suffers a van Dam-Veltman-Zakharov (vDVZ) discontinuity [110, 111]. Generalization of massive theory to curved background was done by Boulware-Deser. This generalization leads to the existence of a typical ghost, the so-called Boulware-Deser ghost [112]. Several models of massive theory were proposed by some authors in order to avoid discontinuity and ghost problems. One of the ghost-free massive theories in three dimensions was introduced by Bergshoeff, Hohm and Townsend (new massive gravity (NMG)) [113]. However, NMG has ghost problem in four and higher dimensions. Therefore, in order to resolve ghost problem in diverse dimensions, a new theory of massive gravity was proposed by de Rham, Gabadadze and Tolley (dRGT) in 2011 [114–116]. The stability of dRGT massive theory was studied and it was shown that such theory enjoys absence of the Boulware-Deser ghost [117, 118]. Neutron stars, black hole and cosmological solutions have been investigated in dRGT massive gravity [119–133]. Also, reentrant phase transitions of higher-dimensional AdS black holes and behavior of quasinormal modes and van der Waals like phase transition of charged AdS black holes in massive gravity have been studied in Refs. [134, 135].

It is notable that in massive gravity theory, the mass terms are produced by consideration of a reference metric. Considering the reference metric in massive gravity, one finds that it plays a crucial role in construction of exact solutions [116]. In this regard, Vegh introduced a new reference metric which was motivated by applications of

gauge/gravity duality [136]. It is believed that the graviton may behave like a lattice and exhibits a Drude peak in this model of massive theory [136]. Another property of this model is related to ghost-free and stability for arbitrary singular metric [137]. The action of massive gravity in an arbitrary d -dimensions is given by

$$\mathcal{I}_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^d x \sqrt{-g} \left[\mathcal{R} + m^2 \sum_{i=1}^4 c_i \mathcal{U}_i(g, f) \right], \quad (1)$$

where \mathcal{R} is the scalar curvature and m^2 is related to the mass of gravitons. In addition, f is a fixed symmetric tensor, c_i 's are some constants, and \mathcal{U}_i 's are symmetric polynomials of the eigenvalues of matrix $\mathcal{K}_\nu^\mu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$ which are as follow

$$\begin{aligned} \mathcal{U}_1 &= [\mathcal{K}], \quad \mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \quad \mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{U}_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]. \end{aligned}$$

Charged black hole solutions with (non)linear field and the existence of van der Waals like behavior in extended phase space and also geometrical thermodynamics by considering dRGT massive gravity have been studied [138–141]. Moreover, the hydrostatic equilibrium equation of neutron stars by using this theory of massive gravity was obtained and it was shown that the maximum mass of neutron stars can be about $3.8M_\odot$ (where M_\odot is mass of the sun) [142]. Also, holographic conductivity in this gravity with PMI field has been investigated in Ref. [143]. Besides, the generalization of this theory to include higher derivative gravity [144, 145] and gravity's rainbow [146, 147] has been done in literatures. In addition, three dimensional (BTZ) charged black hole solutions with (non)linear field have been studied in Ref. [148].

By adding an electromagnetic Lagrangian ($\mathcal{L}(\mathcal{F})$) and the cosmological constant (Λ) to the action (1) with $d = 3$, we have

$$\mathcal{I}_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^3 x \sqrt{-g} \left[\mathcal{R} - 2\Lambda + \mathcal{L}(\mathcal{F}) + m^2 \sum_{i=1}^4 c_i \mathcal{U}_i(g, f) \right]. \quad (2)$$

Varying the action (2) with respect to the gravitational and gauge fields, one can obtain the following field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\mathcal{R} - 2\Lambda) + m^2\chi_{\mu\nu} = T_{\mu\nu}, \quad (3)$$

$$\partial_\mu (\sqrt{-g} \mathcal{L}_{\mathcal{F}} F^{\mu\nu}) = 0, \quad (4)$$

in which $\mathcal{L}_{\mathcal{F}} = d\mathcal{L}(\mathcal{F})/d\mathcal{F}$ where $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor and A_μ is the gauge potential. In addition, $\chi_{\mu\nu}$ is the massive term with the following form

$$\begin{aligned} \chi_{\mu\nu} &= -\frac{c_1}{2}(\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) - \frac{c_2}{2}(\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) - \frac{c_3}{2}(\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 \mathcal{K}_{\mu\nu} + \\ &6\mathcal{U}_1 \mathcal{K}_{\mu\nu}^2 - 6\mathcal{K}_{\mu\nu}^3) - \frac{c_4}{2}(\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 \mathcal{K}_{\mu\nu} + 12\mathcal{U}_2 \mathcal{K}_{\mu\nu}^2 - 24\mathcal{U}_1 \mathcal{K}_{\mu\nu}^3 + 24\mathcal{K}_{\mu\nu}^4), \end{aligned} \quad (5)$$

and the energy-momentum tensor of Eq. (3) is

$$T_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\mathcal{L}(\mathcal{F}) - 2\mathcal{L}_{\mathcal{F}}F_{\mu\lambda}F_\nu^\lambda. \quad (6)$$

Magnetic branes (or horizonless solution) are interesting objects which have been investigated by many authors [149–161]. Our main motivation here is to understand the effects of two generalizations on the magnetic horizonless solutions with interpretation of topological defects. These two generalizations include massive gravity and PMI electromagnetic field. Considering the applications of topological defects in dark matter, CMB, gravitational waves, large scale structure and etc., it is necessary to investigate the effects of the massive gravitons on the structure and formation of topological defects. Here, we intend to show how generalization to massive gravity would modify geometrical structure of the magnetic solutions. To do so, we apply the massive gravity generalization and investigate geometrical properties such as deficit angle. Considering the electromagnetically charged aspect of the objects of interest in this paper (magnetic solutions), we will take two cases of linear and nonlinear electromagnetic fields into account. Here, we would investigate the effects of Maxwell and PMI electromagnetic fields on the deficit angle, hence, geometrical structure of the topological defects known as horizonless magnetic solutions. The combinations of massive gravity and PMI theory is another subject of interest which would be addressed.

II. MAGNETIC SOLUTIONS IN EINSTEIN-MASSIVE GRAVITY WITH MAXWELL FIELD

In this section, we want to obtain the magnetic solutions of Eqs. (3) and (4) by considering the Maxwell electromagnetic field ($\mathcal{L}(\mathcal{F}) = -\mathcal{F}$). To do so, we use the new metric of three dimensional spacetime with $(-++)$ signature which was introduced in Ref. [161]

$$ds^2 = -\frac{\rho^2}{l^2}dt^2 + \frac{d\rho^2}{g(\rho)} + l^2g(\rho)d\varphi^2, \quad (7)$$

where $g(\rho)$ is an arbitrary function of radial coordinate ρ which should be determine. The scale length factor l is related to the cosmological constant Λ , and the angular coordinate φ is dimensionless as usual and ranges in $[0, 2\pi]$. The motivation of considering the metric gauge [$g_{tt} \propto -\rho^2$ and $(g_{\rho\rho})^{-1} \propto g_{\varphi\varphi}$] instead of the usual Schwarzschild like gauge [$(g_{\rho\rho})^{-1} \propto g_{tt}$ and $g_{\varphi\varphi} \propto \rho^2$] comes from the fact that we are looking for magnetic solutions without curvature singularity. It is easy to show that using a suitable transformation, the metric (7) can be mapped to 3-dimensional Schwarzschild like spacetime locally, but not globally [161].

In order to obtain exact solutions, we should make a choice for the reference metric. We consider the following ansatz metric

$$f_{\mu\nu} = \text{diag}\left(\frac{-c^2}{l^2}, 0, 0\right), \quad (8)$$

where in the above equation c is a positive constant. Using the metric ansatz (8), \mathcal{U}_i 's are [138, 148]

$$\mathcal{U}_1 = \frac{c}{\rho}, \quad \mathcal{U}_2 = \mathcal{U}_3 = \mathcal{U}_4 = 0, \quad (9)$$

which indicate that the only contribution of massive gravity comes from \mathcal{U}_1 in three dimensions. Before proceeding we give a reason for such choice of the reference metric (8). For three dimensional black holes, the spacetime metric with $(-, +, +)$ signature has the following explicit form

$$ds^2 = -g(\rho)dt^2 + \frac{d\rho^2}{g(\rho)} + \rho^2d\varphi^2. \quad (10)$$

In order to obtain exact black hole solutions, we consider the ansatz metric as $f_{\mu\nu} = \text{diag}(0, 0, c^2)$ (see Refs. [136], [138] and [140], for more details). Here, the metric function ($g(\rho)$) is factors of radial and spatial coordinates in magnetic spacetime metric (Eq. 7). In order to have exact solutions in an axially symmetric spacetime with the form (7), it is necessary to consider the reference metric as $f_{\mu\nu} = \text{diag}(\frac{-c^2}{l^2}, 0, 0)$. This form of reference metric is expectable. Comparing black hole metric, Eq. (10), with magnetic spacetime, Eq. (7), we find that Eq. (7) can be reproduced from Eq. (10) by the following local transformations:

$$t \longrightarrow il\varphi \quad \& \quad \varphi \longrightarrow it/l. \quad (11)$$

Since we changed the role of t and φ coordinates, the nonzero component of the reference metric should be changed accordingly.

Since we are going to study the linearly magnetic solutions, we choose the Lagrangian of Maxwell field $\mathcal{L}(\mathcal{F}) = -\mathcal{F}$ for Eqs. (2), (4), and (6). It is well-known that the electric field is associated with the time component of the vector potential A_t , while the magnetic field is associated with the angular component A_φ . Due to our interest to investigate the magnetic solutions, we assume the vector potential as

$$A_\mu = h(\rho)\delta_\mu^\varphi. \quad (12)$$

Using the Maxwell equation (4) with $\mathcal{L}(\mathcal{F}) = -\mathcal{F}$, and the metric (7), one finds the following differential equation

$$F_{\varphi\rho} + \rho F'_{\varphi\rho} = 0, \quad (13)$$

where $F_{\varphi\rho} = h'(\rho)$ in which the prime denotes differentiation with respect to ρ . Equation (13) has the following solution

$$F_{\varphi\rho} = \frac{q}{\rho}, \quad (14)$$

where q is an integration constant. To find the metric function $g(\rho)$, one may insert Eq. (14) in the field equation (3) by considering the metric (7). After some calculations, one can obtain the following differential equations

$$\begin{cases} g'(\rho) + 2\Lambda\rho - \frac{2}{\rho} \left(\frac{q}{l}\right)^2 - cc_1 m^2 = 0, & \rho\rho \text{ } (\varphi\varphi) \text{ component} \\ g''(\rho) + 2\Lambda + 2\left(\frac{q}{\rho l}\right)^2 = 0, & tt \text{ component} \end{cases}, \quad (15)$$

where the double prime is the second derivative versus ρ . It is straightforward to show that these equations have the following solution

$$g(\rho) = m_0 - \Lambda\rho^2 + \frac{2q^2}{l^2} \ln\left(\frac{\rho}{l}\right) + cc_1 m^2 \rho, \quad (16)$$

which m_0 is an integration constant which is related to the mass parameter, and l is an arbitrary constant with length dimension which is coming from the fact that the logarithmic arguments should be dimensionless. As you can see, the massive parameter appear in the metric function as a factor for the linear function of ρ . We should note that the obtained metric function (16) satisfies all components of the field equation (3), simultaneously. In addition, the asymptotical behavior of the solution (16) is adS or dS provided $\Lambda < 0$ or $\Lambda > 0$. Also, it is worthwhile to mention that in the absence of massive parameter ($m = 0$), the metric function (16) reduces to the result of Ref. [161] for $s = 1$.

1. Energy Conditions

Now, we examine the energy conditions to find physical solutions. To do so, we consider the orthonormal contravariant basis vectors, and then we obtain the three dimensional energy momentum tensor as $T^{\mu\nu} = \text{diag}(\mu, p_r, p_t)$ in which μ , p_r , and p_t are the energy density, the radial pressure and the tangential pressure, respectively. Having the energy momentum tensor at hand, we are in a position to investigate the energy conditions. We use the following known constraints in three dimensions

$\begin{aligned} p_r + \mu &\geq 0 \\ p_t + \mu &\geq 0 \end{aligned},$	for null energy condition (NEC)
$\begin{aligned} \mu &\geq 0 \\ p_r + \mu &\geq 0 \\ p_t + \mu &\geq 0 \end{aligned},$	for weak energy condition (WEC)
$\begin{aligned} \mu &\geq 0 \\ -\mu \leq p_r \leq \mu \\ -\mu \leq p_t \leq \mu \end{aligned},$	for dominant energy condition (DEC)
$\begin{aligned} p_r + \mu &\geq 0 \\ p_t + \mu &\geq 0 \\ \mu + p_r + p_t &\geq 0 \end{aligned},$	for strong energy condition (SEC)

Table (1): Energy conditions criteria

In order to simplify the mathematics and physical interpretations, we use the following orthonormal contravariant (hatted) basis vectors for diagonal static metric (7)

$$\mathbf{e}_{\hat{t}} = \frac{l}{\rho} \frac{\partial}{\partial t}, \quad \mathbf{e}_{\hat{\rho}} = \sqrt{g} \frac{\partial}{\partial \rho}, \quad \mathbf{e}_{\hat{\phi}} = \frac{1}{l\sqrt{g}} \frac{\partial}{\partial \phi}. \quad (17)$$

It is a matter of straightforward calculations to show that the nonzero components of stress-energy tensor are

$$T^{\hat{t}\hat{t}} = T^{\hat{\rho}\hat{\rho}} = T^{\hat{\phi}\hat{\phi}} = \left(\frac{F_{\phi\rho}}{l}\right)^2. \quad (18)$$

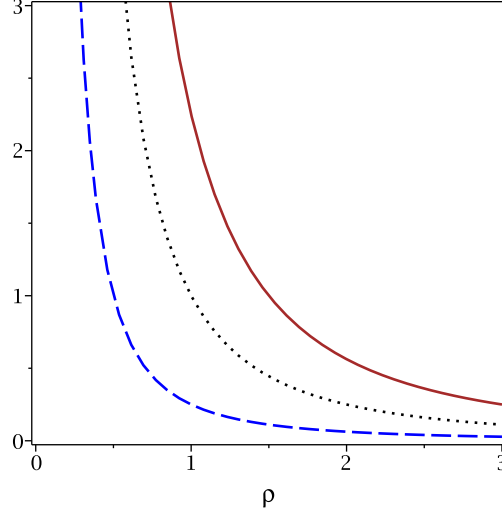


FIG. 1: $T^{\hat{t}\hat{t}}$ versus ρ for $l = 1$ and $q = 0.5$ (dashed line), $q = 1.0$ (dotted line) and $q = 1.5$ (continuous line).

All components of stress-energy tensor are the same and positive and it is easy to find that NEC, WEC, DEC and SEC are satisfied, simultaneously.

As one can see, the massive parameter do not contribute to the energy-momentum tensor, so the energy conditions are independent of the massive parameter. In order to investigation the effects of charge on the energy density of the spacetime, we plot the $T^{\hat{t}\hat{t}}$ versus ρ . Considering Fig. 1, one can find that the energy density of the spacetime is positive everywhere, and increasing the charge parameter leads to increasing the concentration of energy density.

2. Geometric Properties

Now, we want to study the properties of spacetime described by Eq. (7) with obtained metric function (16). At first, we calculate $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ for examination of existence of curvature singularity

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = 12\Lambda^2 - \frac{8\Lambda q^2}{l^2\rho^2} + \frac{2cc_1m^2(4q^2 + cc_1m^2l^2\rho - 4\Lambda l^2\rho^2)}{l^2\rho^3} + \frac{12q^4}{l^4\rho^4}. \quad (19)$$

Considering Eq. (19), the Kretschmann scalar reduces to $12\Lambda^2$ for $\rho \rightarrow \infty$, which confirms that the asymptotical behavior of this spacetime is (a)dS. It is also obvious that the Kretschmann scalar diverges at $\rho = 0$, and therefore one might think that there is a curvature singularity located at $\rho = 0$. But as we will see, the spacetime will never achieve $\rho = 0$. There are two possible cases for the metric function: the metric function has no real positive root which is interpreted as naked singularity (this case is not of interest here), or metric function has at least one real positive root. If one considers r_0 as the largest root of metric function, it is clear that for $\rho < r_0$ there will be a change in signature of metric (see Fig. 2). In other words, for $\rho < r_0$ the metric function is negative, hence metric signature is $(-, -, -)$, and for $\rho > r_0$ the metric function is positive, therefore metric signature is legal $(-, +, +)$. This change in the metric signature results into a conclusion: it is not possible to extend spacetime to $\rho < r_0$. In order to exclude the forbidden zone ($\rho < r_0$), we introduce a new radial coordinate r as

$$r^2 = \rho^2 - r_0^2 \implies d\rho^2 = \frac{r^2}{r^2 + r_0^2} dr^2, \quad (20)$$

where for the allowed region, $\rho \geq r_0$, leads to $r \geq 0$ in the new coordinate system. Applying this coordinate transformation, the metric (7) should be written as

$$ds^2 = -\frac{r^2 + r_0^2}{l^2} dt^2 + \frac{r^2}{(r^2 + r_0^2)g(r)} dr^2 + l^2 g(r) d\varphi^2, \quad (21)$$

in which the coordinate r assumes the values $0 \leq r < \infty$, and obtained $g(r)$ (Eq. (16)) is now given by

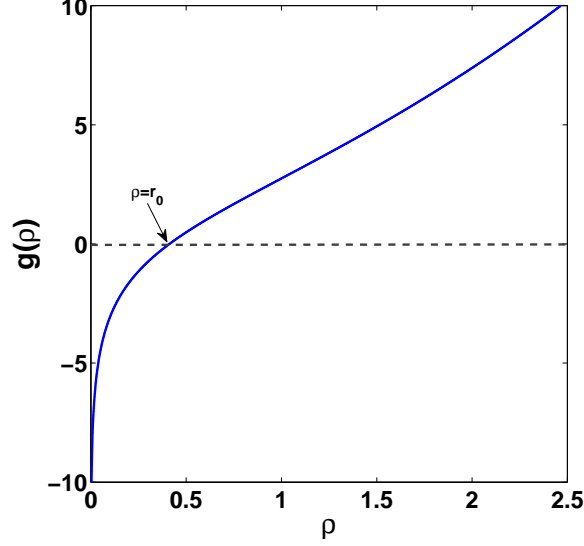


FIG. 2: $g(\rho)$ versus ρ for $l = 1$, $q = 1$, $\Lambda = -1$, $c = 1$, $c_1 = 1$, $m = 0.5$ and $m_0 = 1.5$.

$$g(r) = m_0 - \Lambda (r^2 + r_0^2) + \frac{q^2}{l^2} \ln \left(\frac{r^2 + r_0^2}{l^2} \right) + cc_1 m^2 \sqrt{r^2 + r_0^2}. \quad (22)$$

The nonzero component of electromagnetic field in the new coordinate can be given by

$$F_{\varphi r} = \frac{q}{\sqrt{r^2 + r_0^2}}, \quad (23)$$

One can show that all curvature invariants do not diverge in the range $0 \leq r < \infty$, and $g(r)$ (Eq. (22)) is positive definite for $0 \leq r < \infty$. It is evident that for having singular solutions both r and r_0 must be zero whereas this case is never reached due to considering nonzero value for r_0 . So, this spacetime has no curvature singularity and horizon. Due to the fact that the limit of the ratio "circumference/radius" is not 2π , the spacetime (21) has a conic geometry and therefore the spacetime has a conical singularity at $r = 0$

$$\lim_{r \rightarrow 0} \frac{1}{r} \sqrt{\frac{g_{\varphi\varphi}}{g_{rr}}} \neq 1. \quad (24)$$

On the other hand, the conical singularity can be removed if one exchanges the coordinate φ with the following period

$$Period_\varphi = 2\pi \left(\lim_{r \rightarrow 0} \frac{1}{r} \sqrt{\frac{g_{\varphi\varphi}}{g_{rr}}} \right)^{-1} = 2\pi (1 - 4\mu), \quad (25)$$

in which the deficit angle define as $\delta\varphi = 8\pi\mu$, where μ is given by

$$\mu = \frac{1}{4} + \frac{1}{4lr_0\Omega}, \quad (26)$$

where Ω is

$$\Omega = \Lambda - \frac{q^2}{l^2 r_0^2} - \frac{cc_1 m^2}{2r_0}. \quad (27)$$

In order to have a better insight of the behavior of deficit angle, we calculate the root and divergence points of the deficit angle as

$$r_0|_{\delta\varphi=0} = \begin{cases} \frac{1}{4l\Lambda} \left(cc_1 m^2 l - 2 \pm \sqrt{c^2 c_1^2 m^4 l^2 - 4(cc_1 m^2 l - 1 - 4\Lambda q^2)} \right) \\ \frac{1}{4l\Lambda} \left(-cc_1 m^2 l - 2 \pm \sqrt{c^2 c_1^2 m^4 l^2 - 4(cc_1 m^2 l - 1 - 4\Lambda q^2)} \right) \end{cases}, \quad (28)$$

$$r_0|_{\delta\varphi \rightarrow \infty} = \begin{cases} \frac{1}{4\Lambda} \left(cc_1 m^2 l \pm \sqrt{c^2 c_1^2 m^4 l^2 + 16\Lambda q^2} \right) \\ \frac{1}{4\Lambda} \left(-cc_1 m^2 l \pm \sqrt{c^2 c_1^2 m^4 l^2 + 16\Lambda q^2} \right) \end{cases}. \quad (29)$$

Here, we see that the roots are functions of the cosmological constant, massive gravity and electric charge. Existence of the real valued root is restricted to following condition

$$c^2 c_1^2 m^4 l^2 - 4 (cc_1 m^2 l - 1 - 4\Lambda q^2) \geq 0. \quad (30)$$

The effects of the massive gravity and electric charge are only observed in numerator of the roots while the effects of the cosmological constant could be observed in both numerator and denominator of the roots. The electric charge is coupled with cosmological constant. While such coupling is not observed for the massive gravity.

As for the divergencies of the deficit angle, one can observe that its existence is also restricted to satisfaction of specific condition in the following form

$$c^2 c_1^2 m^4 l^2 + 16\Lambda q^2 \geq 0 \quad (31)$$

In the absence of the massive gravity, only for dS spacetime divergencies are observable for deficit angle. Generalization to massive gravity provides the possibility of the divergencies for deficit angle in adS spacetime under certain circumstances. This highlights the effects of the massive gravity. Here, similar to the case of roots, a coupling between cosmological constant and electric charge is observed while such coupling could not be seen for massive gravity.

III. GENERALIZATION OF ACHIEVEMENTS TO THE CASE OF NONLINEAR ELECTRODYNAMICS: PMI THEORY

In this section, we are going to obtain the solutions in presence of PMI source and investigate the properties. We start with the following PMI Lagrangian

$$\mathcal{L}_{PMI}(\mathcal{F}) = (-\kappa \mathcal{F})^s, \quad (32)$$

where κ and s are coupling and power constants, respectively. Obviously, the PMI Lagrangian (32) reduces to the standard Maxwell Lagrangian ($\mathcal{L}_{Maxwell}(\mathcal{F}) = -\mathcal{F}$) for $s = 1$ and $\kappa = 1$ which we have investigated before.

Following the method of section II and considering Eqs. (4), (7) and (32), one can obtain the following differential equation for nonzero component of Faraday tensor

$$F_{\varphi\rho} + (2s - 1)\rho F'_{\varphi\rho} = 0, \quad (33)$$

with the following solution

$$F_{\varphi\rho} = q\rho^{1/(1-2s)}, \quad (34)$$

in which q is an integration constant. In order to have a physical asymptotical behavior, we should consider $s > 1/2$. On the other hand, one can easily show that the vector potential A_φ , is

$$A_\varphi = q\rho^{\frac{2(s-1)}{2s-1}}, \quad (35)$$

the electromagnetic gauge potential should be finite at infinity ($\rho \rightarrow \infty$), therefore, one should impose following restriction to have this property, so we have

$$\frac{2(s-1)}{2s-1} < 0. \quad (36)$$

The above equation leads to the following restriction on the range of s , as

$$\frac{1}{2} < s < 1. \quad (37)$$

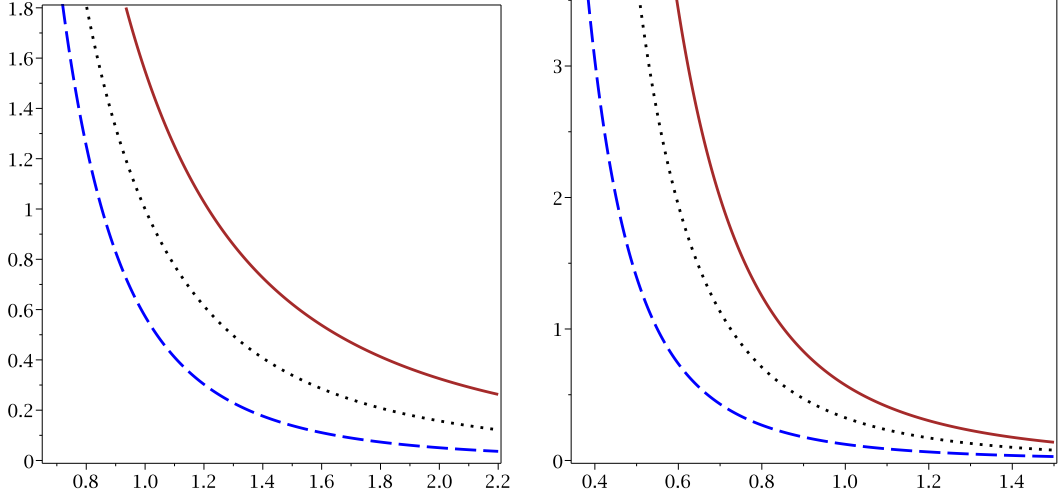


FIG. 3: $T_{\tilde{t}\tilde{t}}$ versus ρ for $l = 1$.

Left diagram: for $q = 1.5$, $s = 0.7$ (dashed line), $s = 0.8$ (dotted line) and $s = 0.9$ (continuous line).

Right diagram: for $s = 0.7$, $q = 0.5$ (dashed line), $q = 1.0$ (dotted line) and $q = 1.5$ (continuous line).

Here, one can insert Eq. (34) in the gravitational field equation (3) by considering the metric (7) to obtain the metric function $g(\rho)$ as

$$g(\rho) = m_0 - \Lambda\rho^2 + cc_1m^2\rho + \frac{(2s-1)^2\rho^2}{2(s-1)}\chi(\rho), \quad (38)$$

where

$$\chi(\rho) = \left(\frac{2q^2}{l^2} \rho^{2/(1-2s)} \right)^s. \quad (39)$$

It is notable that, the obtained metric function in Eq. (38) is related to $s \neq 1$. Also, m_0 is an integration constant which is related to the mass of solutions.

Now, one can calculate the nonzero components of stress-energy tensor by using the introduced basis vectors in Eq. (17) as

$$T_{\tilde{t}\tilde{t}} = \frac{1}{2} \left(\frac{2F_{\phi\rho}^2}{l^2} \right)^s, \quad (40)$$

$$T_{\hat{\rho}\hat{\rho}} = T_{\hat{\phi}\hat{\phi}} = \left(s - \frac{1}{2} \right) \left(\frac{2F_{\phi\rho}^2}{l^2} \right)^s. \quad (41)$$

According to the above equation, $\mu(T_{\tilde{t}\tilde{t}})$ is positive, and so the NEC, WEC, and SEC are satisfied, simultaneously. In addition, in order to satisfy the DEC, the parameter of PMI (s) must be in the range $\frac{1}{2} < s < 1$. As we have mentioned before, the energy conditions do not depend on the massive parameter. Here, we want to investigate the effects of PMI parameter (s) and electrical charge (q) on the energy conditions, so we plot $T_{\tilde{t}\tilde{t}}$ versus ρ in Fig. 3. As one can see, increasing the parameter of PMI theory and electrical charge leads to increasing the concentration of energy density.

One can show that the metric (7) with the metric function (38) has a singularity at $\rho = 0$ by calculating the Kretschmann scalar as

$$\begin{aligned} R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} &= 12\Lambda^2 - \frac{4cc_1m^2}{\rho} \left[2\Lambda - (2s-1)\chi_{q,\rho,s} - \frac{cc_1m^2}{2\rho} \right] \\ &\quad + [(8s^2 - 8s + 3)\chi_{q,\rho,s} - 4(4s-3)\Lambda] \chi_{q,\rho,s}. \end{aligned} \quad (42)$$

From Eq. (42), it is obvious that the Kretschmann scalar reduces to $12\Lambda^2$ for $\rho \rightarrow \infty$ and diverges at $\rho = 0$. On the other hand, as we mentioned before, it is not possible to extend spacetime to $\rho < r_0$ because of signature changing. Also, one can apply the coordinate transformation (20) to the metric (7) and find the metric function as

$$g(r) = m_0 - \Lambda (r^2 + r_0^2) + cc_1 m^2 (r^2 + r_0^2)^{1/2} + \frac{(2s-1)^2 (r^2 + r_0^2)}{2(s-1)} \chi(r), \quad (43)$$

where

$$\chi(r) = \left(\frac{2q^2}{l^2} (r^2 + r_0^2)^{1/(1-2s)} \right)^s, \quad (44)$$

and the electromagnetic field in the new coordinate is

$$F_{\varphi r} = q (r^2 + r_0^2)^{1/2(1-2s)}. \quad (45)$$

Since all curvature invariants do not diverge in the range $0 \leq r < \infty$, one finds that there is no essential singularity. But like the Maxwell case, this spacetime has a conical singularity at $r = 0$ with the deficit angle $\delta\varphi = 8\pi\mu$ where μ is given by Eq. (26) and Ω has the following form

$$\Omega = \Lambda - \frac{cc_1 m^2}{2r_0} - \frac{(2s-1)}{2} \left(\frac{2q^2}{l^2} r_0^{2/(1-2s)} \right)^s. \quad (46)$$

Due to complexity of obtained relation in Eq. (46), it is not possible to calculate the root and divergence points of deficit angle analytically, therefore, we study them in some graphs in next section.

IV. DEFICIT ANGLE DIAGRAMS

In order to study the effects of different parameters on the properties of deficit angle for the Maxwell and PMI cases, we have plotted various diagrams (Figs. 4-6 for Maxwell case and Figs. 7-10 for PMI case). The left panels are dedicated to adS spacetime while the right ones are related to dS spacetime. In Ref. [162], it was pointed out that in order to remove ensemble dependency, l should be replaced by following relation

$$\Lambda(\varepsilon) = \pm \frac{1}{l(\varepsilon)}, \quad (47)$$

where the positive branch is related to dS spacetime and the opposite is for adS solutions. Hereafter, we employ Eq. (47) to plot deficit angle diagrams. It is notable to highlight a few remarks regarding to values of deficit angle. The deficit angle is restricted by an upper limit provided by geometrical properties of the solutions. Its value could not exceed 2π , and more precisely, deficit angle could have values in range of $-\infty < \delta\varphi \leq 2\pi$.

Depending on the choices of different parameters, the deficit angle of Maxwell-adS and PMI-adS solutions could have a minimum. In adS case, except for neutral solutions, the deficit angle could have; I) two roots with one region of negativity located between these two roots. II) One extreme root located at the minimum with deficit angle being only positive. III) No root and deficit angle is always positive. The minimum is an increasing function of the massive parameter (left panels of Figs. 4 and 7), electric charge (left panels of Fig. 5 and 8) and c_1 (left panels of Fig. 6 and 9). By considering negative values for c_1 , it is possible to have one of the following cases; I) one divergency located between two roots. II) Two divergency which are located between two roots. Between the divergencies, the deficit angle is positive but its value is out of the permitted values. In these two cases, the positive deficit angle could only be observed before smaller root and after larger root. Interestingly, in the absence of electric charge, the deficit angle is only an increasing function of r_0 (left panels of Figs. 5 and 8).

For the Maxwell-dS and PMI-dS spacetimes, interestingly, only one root and divergency are observed. The root and divergency are increasing functions of the massive gravity (right panels of Fig. 4 and 7), electric charge (right panels of Fig. 5 and 8) and c_1 (right panels of Fig. 6 and 9). The divergency is located after root. The deficit angle is only positive before root. After divergency, the deficit angle is positive but its values are not in permitted region. The only exception is for the absence of electric charge (right panels of Fig. 5 and 8). In this case, no root is observed and deficit angle is negative valued.

In the case of PMI theory, another free parameter (nonlinearity parameter) exists. Evidently, the minimum in adS case is an increasing function of this parameter (left panel of Fig. 10). For dS spacetime, the root and divergency are increasing functions of this parameter.

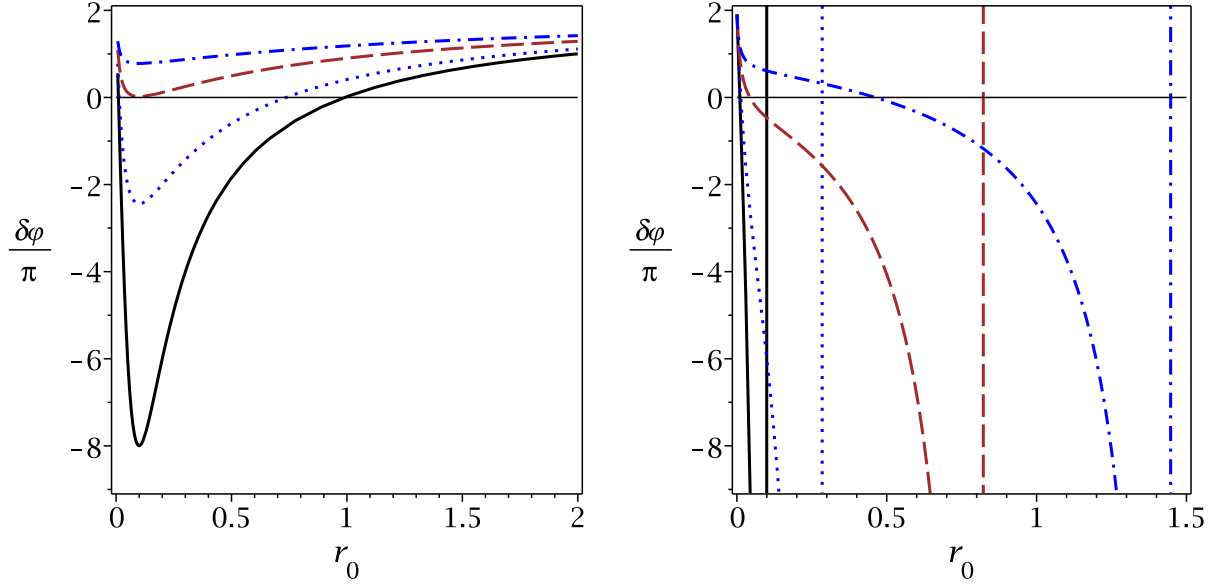


FIG. 4: **Maxwell solutions:** $\delta\varphi$ versus r_0 for $q = 0.1$, $c = 1$ and $c_1 = 2$, $m = 0$ (continuous line), $m = 0.5$ (dotted line), $m = 0.9$ (dashed line) and $m = 1.2$ (dashed-dotted line).
Left diagram: $\Lambda = -1$; **Right diagram:** $\Lambda = 1$.

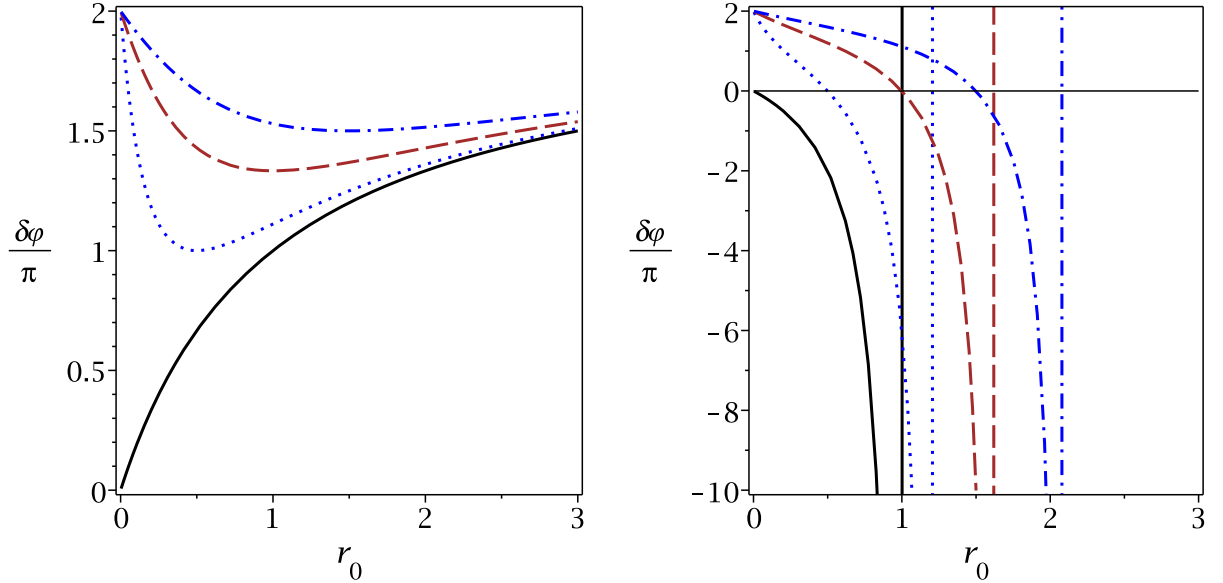


FIG. 5: **Maxwell solutions:** $\delta\varphi$ versus r_0 for $m = 1$, $c = 1$ and $c_1 = 2$, $q = 0$ (continuous line), $q = 0.5$ (dotted line), $q = 1$ (dashed line) and $q = 1.5$ (dashed-dotted line).
Left diagram: $\Lambda = -1$; **Right diagram:** $\Lambda = 1$.

Depending on values of deficit angle, the geometrical structure of the magnetic solutions will be determined. Our solutions contain a conical singularity. This conical singularity is built by considering a 2-dimensional plane replaced with cutting an arbitrary slice and sewing together the edges. The singular point is located at the apex of cone. Now, considering this concept, one can see that positive values of the deficit angle represent missing segment of the 2-dimensional plane. On the contrary, the negative values of the deficit angle represent the additional part that we can add to the mentioned plane. Therefore, the positivity/negativity of the deficit angle plays a crucial role in the topological structure of the solutions. Here, we see that depending on choices of different parameters, it is possible to obtain negative and positive values of the deficit angle. The roots of deficit angle could be interpreted as

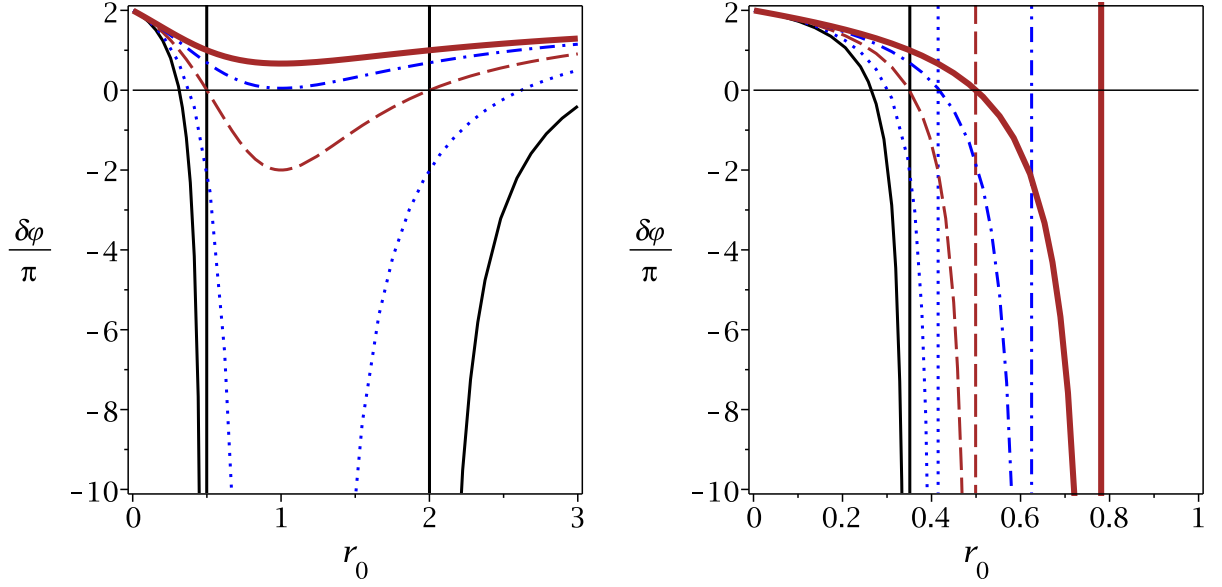


FIG. 6: **Maxwell solutions:** $\delta\varphi$ versus r_0 for $q = 0.1$, $c = 1$ and $m = 1$, $c_1 = -5$ (continuous line), $c_1 = -4$ (dotted line), $c_1 = -3$ (dashed line), $c_1 = -1.95$ (dashed-dotted line) and $c_1 = -1$ (bold line).
Left diagram: $\Lambda = -1$; **Right diagram:** $\Lambda = 1$.

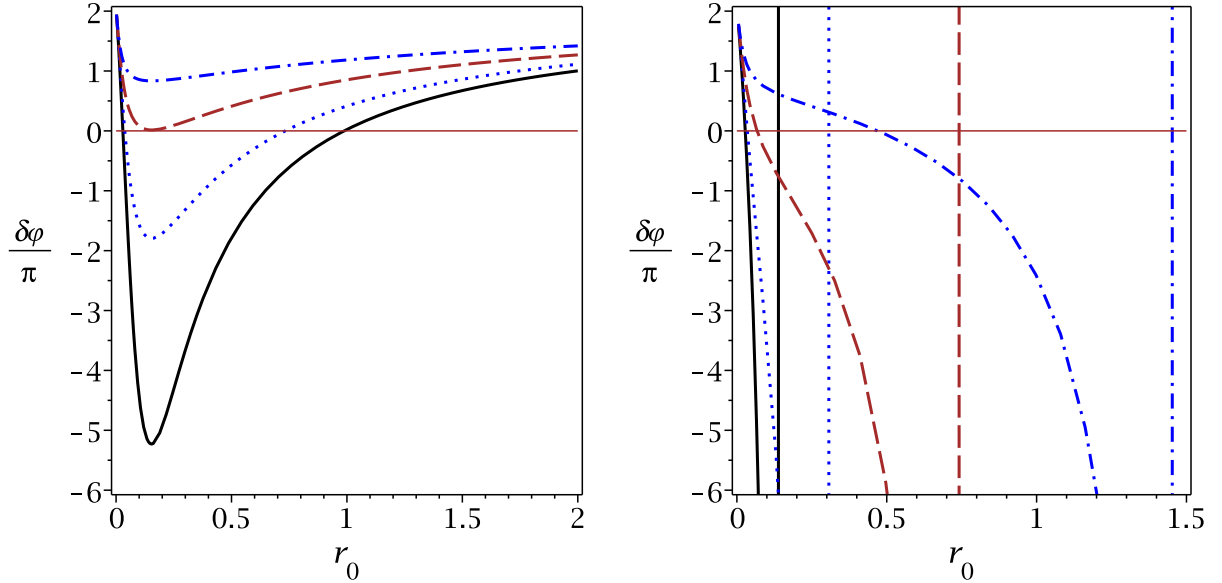


FIG. 7: **PMI solutions:** $\delta\varphi$ versus r_0 for $q = 0.1$, $c = 1$, $c_1 = 2$ and $s = 0.9$, $m = 0$ (continuous line), $m = 0.5$ (dotted line), $m = 0.85$ (dashed line) and $m = 1.2$ (dashed-dotted line).
Left diagram: $\Lambda = -1$; **Right diagram:** $\Lambda = 1$.

transition points in which the total shape of the object is modified. On the other hand, the existence of divergencies for deficit angle marks the possibility of the absence of magnetic solutions which was observed for both the dS and adS spacetimes. For more details regarding the negative deficit angle, we refer the reader to Refs [163–167]. The main motivation of this paper was understanding the effects of massive gravity and PMI theory on the magnetic solutions. The variation in deficit angle shows that the total structure of the solutions depends on contributions of these two generalizations. Specially, we observed that generalization to massive gravity provided the possibility of existence of divergence points for adS spacetime. It is worthwhile to mention that for adS case, between two divergencies, the values of deficit angle are within prohibited range. This indicates that there is no acceptable deficit angle between

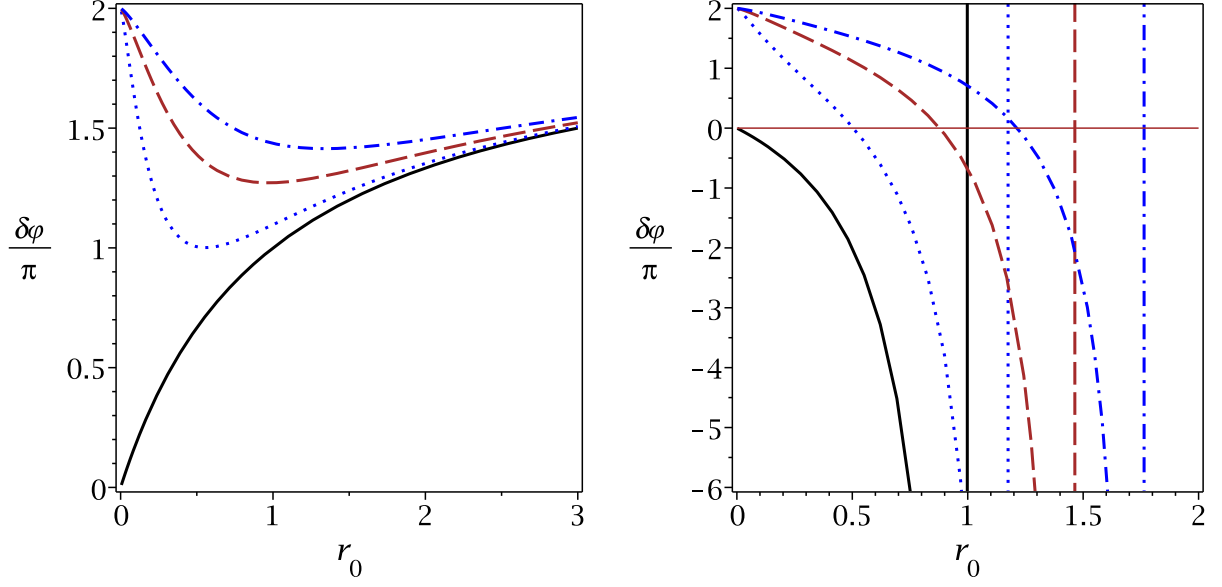


FIG. 8: **PMI solutions:** $\delta\varphi$ versus r_0 for $m = 1$, $c = 1$, $c_1 = 2$ and $s = 0.9$, $q = 0$ (continuous line), $q = 0.5$ (dotted line), $q = 1$ (dashed line) and $q = 1.5$ (dashed-dotted line).
Left diagram: $\Lambda = -1$; **Right diagram:** $\Lambda = 1$.

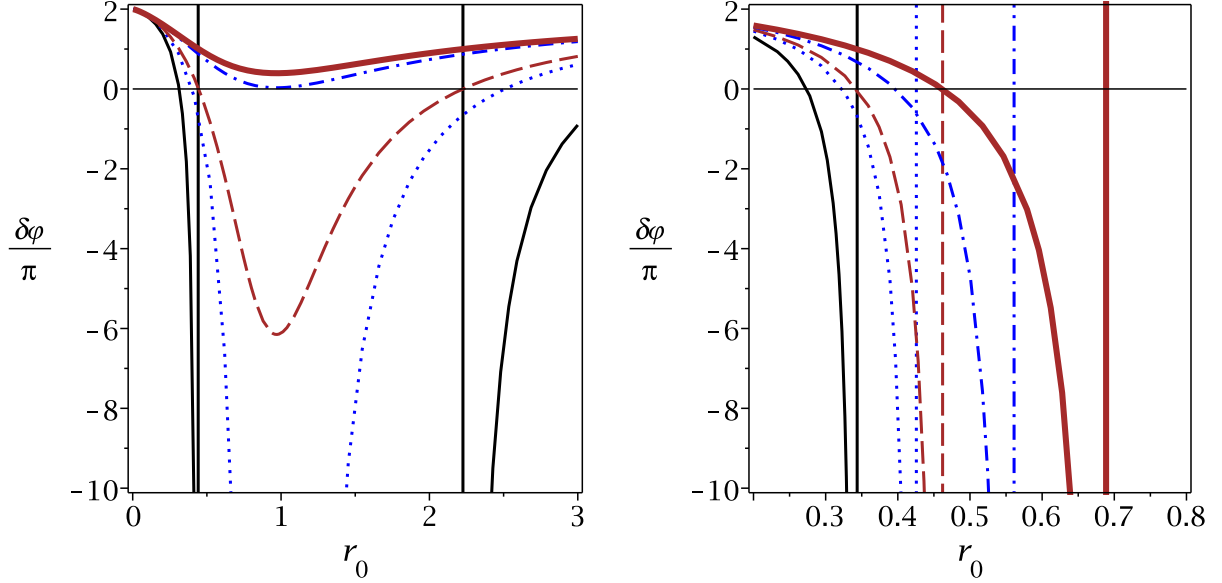


FIG. 9: **PMI solutions:** $\delta\varphi$ versus r_0 for $q = 0.1$, $c = 1$, $m = 1$ and $s = 0.9$, $c_1 = -5$ (continuous line), $c_1 = -3.49$ (dotted line), $c_1 = -3$ (dashed line), $c_1 = -1.46$ (dashed-dotted line) and $c_1 = -1$ (bold line).
Left diagram: $\Lambda = -1$; **Right diagram:** $\Lambda = 1$.

the divergencies in adS case.

V. CONCLUSIONS

In this paper, we have considered magnetic solutions which contain a conical singularity without any event horizon and curvature singularity. The set up for the gravity and energy momentum tensor were consideration of two generalizations: massive gravity and PMI nonlinear electromagnetic field.

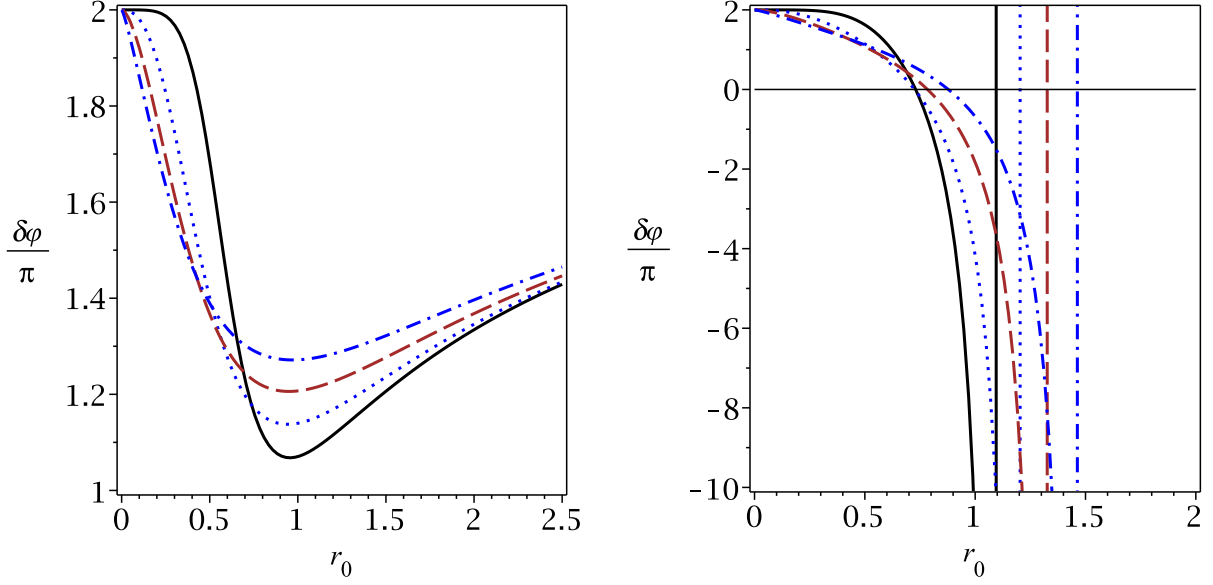


FIG. 10: **PMI solutions:** $\delta\varphi$ versus r_0 for $q = 0.1$, $c = 1$, $c_1 = 2$ and $m = 1$, $s = 0.6$ (continuous line), $s = 0.7$ (dotted line), $s = 0.8$ (dashed line) and $s = 0.9$ (dashed-dotted line).

Left diagram: $\Lambda = -1$; **Right diagram:** $\Lambda = 1$.

The geometrical properties of the solutions were obtained and deficit angle for the two cases of Maxwell-massive and PMI-massive were extracted. It was shown that the general structure of the solutions depends on choices of different parameters through positivity and negativity of the deficit angle. Existence of root and divergency were reported and it was shown that these properties of the solutions depend on the choices of different parameter, such as massive gravity and nonlinearity parameter. In addition, it was shown that depending on the nature of background (being dS or adS), deficit angle, hence geometrical structure of the solutions would be different. The difference was highlighted analytically and numerically through several diagrams.

The existence of root and divergency for deficit angle was reported which indicates that under certain condonations, suitable choices of different parameters, topological defects known as magnetic solutions would enjoy geometrical phase transition. The dependency of geometrical phase transition on nonlinearity parameter and massive gravity highlighted the importance and roles of massive gravity and also nonlinear electromagnetic field generalizations. Especially, the existence of divergency for adS spacetime in the presence of massive gravity could be pointed out.

Acknowledgments

The authors wish to thank Shiraz University Research Council. This work has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha.

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